Chapter 6 Free-Response Practice Test

Directions: This practice test features free-response questions based on the content in Chapter 6: Integration Techniques.

6.1: Integration by Parts
6.2: Trigonometric Integrals
6.3: Trigonometric Substitution
6.4: Integration by Partial Fractions
6.5: Improper Integrals

For each question, show your work. If you encounter difficulties with a question, then move on and return to it later. Follow these guidelines:

- Do not use a calculator of any kind. All of these problems are designed to contain simple numbers.
- Adhere to the time limit of 90 minutes.
- After you complete all the questions, score yourself according to the Solutions document. Note any topics that require revision.

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Integration Techniques

Number of Questions—12

Suggested Time—1 hour 30 minutes

NO CALCULATOR

Scoring Chart

Section	Points Earned	Points Available
Short Integrals		40
Question 9		15
Question 10		15
Question 11		15
Question 12		15
TOTAL		**100**

Short Integrals

1. $\displaystyle\int (3t+2)\cos 5t\,\mathrm{d}t$ *(5 pts.)*

2. $\displaystyle\int_0^{\pi/6} \sin^2\theta\cos^3\theta\,\mathrm{d}\theta$ *(5 pts.)*

3. $\displaystyle\int \sin(5\theta)\cos(9\theta)\,\mathrm{d}\theta$ *(5 pts.)*

4. $\displaystyle\int_{-\infty}^{\infty} \frac{2}{x^2+16}\,\mathrm{d}x$ *(5 pts.)*

5. $\displaystyle\int x^2 e^{-x/4}\,\mathrm{d}x$ *(5 pts.)*

6. $\displaystyle\int_5^{10} \frac{x^3}{5\sqrt{x^2-25}}\,\mathrm{d}x$ *(5 pts.)*

7. $\displaystyle\int \frac{x+2}{\sqrt{30-8x-2x^2}}\,\mathrm{d}x$ *(5 pts.)*

8. $\displaystyle\int_1^4 e^{\sqrt{x}}\,\mathrm{d}x$ *(5 pts.)*

Long Questions

9. This question examines the family of integrals $\int \sec^m x \tan^n x \, \mathrm{d}x$.

(a) Evaluate $\int \sec^m x \tan^n x \, \mathrm{d}x$ for $m = 3$ and $n = 5$. *(5 pts.)*

(b) Evaluate $\int \sec^m x \tan^n x \, \mathrm{d}x$ for $m = 4$ and $n = 2$. *(5 pts.)*

(c) Using a trigonometric substitution, show that $\int \frac{x^5\sqrt{x^2+4}}{128} \, \mathrm{d}x$ can be transformed into $\int \sec^3\theta \tan^5\theta \, \mathrm{d}\theta$. *(5 pts.)*

10. Evaluate $\displaystyle\int_0^1 \frac{2x^2}{x^4-81}\,\mathrm{d}x$. *(15 pts.)*

11. Consider the family of integrals $\int_{-1}^{0} \frac{1-x}{2x^2+kx+6}\,\mathrm{d}x$ for some constant k.

(a) Calculate $\int_{-1}^{0} \frac{1-x}{2x^2+kx+6}\,\mathrm{d}x$ for $k=-4$. *(4 pts.)*

(b) Calculate $\int_{-1}^{0} \frac{1-x}{2x^2+kx+6}\,\mathrm{d}x$ for $k=13$. *(6 pts.)*

(c) With $k=-4$, consider the integral $\int_{-1}^{0} \frac{(1-x)^3}{2x^2+kx+6}\,\mathrm{d}x$. Using the substitution $x-1=\sqrt{2}\tan\theta$, convert the integral to be solely in terms of θ. *(5 pts.)*

12. Consider the family of improper integrals $I = \int_1^\infty \frac{1}{x^{3c+6}}\,\mathrm{d}x$, where c is a constant.

(a) Find all values of c for which the integral converges. (2 pts.)

(b) For $c = -\frac{1}{3}$, show that $I = \frac{1}{4}$. (4 pts.)

(c) Does $\int_1^\infty \frac{|\cos x|}{x^5}\,\mathrm{d}x$ converge or diverge? (3 pts.)

(d) For $c = -\dfrac{17}{9}$, evaluate $\displaystyle\int_0^1 \frac{1}{x^{3c+6}}\,\mathrm{d}x$ or show that it diverges. (6 pts.)

This marks the end of the test. The solutions and scoring rubric begin on the next page.

Short Questions (5 points each)

1. We use Integration by Parts with

$$u = 3t+2 \qquad \text{and} \qquad \mathrm{d}v = \cos 5t\,\mathrm{d}t\,.$$ *

Then

$$\mathrm{d}u = 3\,\mathrm{d}t \qquad \text{and} \qquad v = \tfrac{1}{5}\sin 5t\,.$$ *

Thus,

$$\begin{aligned}\int (3t+2)\cos 5t\,\mathrm{d}t &= (3t+2)\left(\tfrac{1}{5}\sin 5t\right) - \int \left(\tfrac{1}{5}\sin 5t\right)(3)\,\mathrm{d}t && * \\ &= \boxed{\tfrac{1}{5}(3t+2)\sin 5t + \tfrac{3}{25}\cos 5t + C} && **\end{aligned}$$

2. Because the power of cosine is odd, we can save an extra factor $\cos\theta$ and convert the remaining terms to be in terms of sine:

$$\begin{aligned}\int_0^{\pi/6} \sin^2\theta\,\cos^3\theta\,\mathrm{d}\theta &= \int_0^{\pi/6} \sin^2\theta\,\cos^2\theta\,\cos\theta\,\mathrm{d}\theta \\ &= \int_0^{\pi/6} \sin^2\theta\,(1-\sin^2\theta)\cos\theta\,\mathrm{d}\theta\,. && *\end{aligned}$$

To strip the extra factor $\cos\theta$, we let

$$u = \sin\theta \implies \mathrm{d}u = \cos\theta\,\mathrm{d}\theta\,.$$ *

When $\theta = 0$, $u = 0$; when $\theta = \dfrac{\pi}{6}$, $u = \dfrac{1}{2}$. The integral therefore becomes

$$\begin{aligned}\int_0^{1/2} [u^2(1-u^2)]\,\mathrm{d}u &= \int_0^{1/2} (u^2-u^4)\,\mathrm{d}u && * \\ &= \left(\tfrac{1}{3}u^3 - \tfrac{1}{5}u^5\right)\Big|_0^{1/2} && * \\ &= \boxed{\tfrac{17}{480}} && *\end{aligned}$$

3. Using the product-to-sum trigonometric identities,

$$\begin{aligned}\sin(5\theta)\cos(9\theta) &= \tfrac{1}{2}\sin(5\theta-9\theta)+\tfrac{1}{2}\sin(5\theta+9\theta) && * \\ &= \tfrac{1}{2}\sin(-4\theta)+\tfrac{1}{2}\sin(14\theta). && *\end{aligned}$$

Thus,

$$\begin{aligned}\int \sin(5\theta)\cos(9\theta)\,\mathrm{d}\theta &= \int \left[\tfrac{1}{2}\sin(-4\theta)+\tfrac{1}{2}\sin(14\theta)\right]\mathrm{d}\theta \\ &= \boxed{\tfrac{1}{8}\cos(-4\theta)-\tfrac{1}{28}\cos(14\theta)+C} && ***\end{aligned}$$

4. We rewrite the integral as

$$\frac{1}{16}\int_{-\infty}^{\infty}\frac{2}{\frac{x^2}{16}+1}\,\mathrm{d}x = \frac{1}{8}\int_{-\infty}^{\infty}\frac{1}{\left(\frac{x}{4}\right)^2+1}\,\mathrm{d}x. \qquad *$$

We substitute

$$u=\frac{x}{4} \implies \mathrm{d}u=\frac{\mathrm{d}x}{4}. \qquad *$$

In terms of u, the bounds stay at $-\infty$ and ∞. Hence, the integral becomes

$$\frac{1}{2}\int_{-\infty}^{\infty}\frac{1}{u^2+1}\,\mathrm{d}u. \qquad *$$

We can split the integral at any value—say, $u=0$:

$$\frac{1}{2}\int_{-\infty}^{\infty}\frac{1}{u^2+1}\,\mathrm{d}u = \frac{1}{2}\int_{-\infty}^{0}\frac{1}{u^2+1}\,\mathrm{d}u + \frac{1}{2}\int_{0}^{\infty}\frac{1}{u^2+1}\,\mathrm{d}u. \qquad *$$

The first integral is

$$\begin{aligned}\frac{1}{2}\int_{-\infty}^{0}\frac{1}{u^2+1}\,\mathrm{d}u &= \tfrac{1}{2}\lim_{t\to-\infty}\int_{t}^{0}\frac{1}{u^2+1}\,\mathrm{d}u \\ &= \tfrac{1}{2}\lim_{t\to-\infty}\tan^{-1}u\Big|_{t}^{0} \\ &= 0-\tfrac{1}{2}\lim_{t\to-\infty}\tan^{-1}t \\ &= \frac{\pi}{4}.\end{aligned}$$

The second integral is

$$\begin{aligned}\frac{1}{2}\int_0^{\infty}\frac{1}{u^2+1}\,\mathrm{d}u &= \tfrac{1}{2}\lim_{t\to\infty}\int_0^t \frac{1}{u^2+1}\,\mathrm{d}u\\ &= \tfrac{1}{2}\lim_{t\to\infty}\tan^{-1}u\Big|_0^t\\ &= \tfrac{1}{2}\lim_{t\to\infty}\tan^{-1}t-0\\ &= \frac{\pi}{4}.\end{aligned}$$

Hence,

$$\frac{1}{2}\int_{-\infty}^{\infty}\frac{1}{u^2+1}\,\mathrm{d}u = \frac{\pi}{4}+\frac{\pi}{4} = \boxed{\frac{\pi}{2}}$$ *

5. We apply Integration by Parts with

$$u = x^2 \qquad \text{and} \qquad \mathrm{d}v = e^{-x/4}\,\mathrm{d}x.$$ *

Then

$$\mathrm{d}u = 2x\,\mathrm{d}x \qquad \text{and} \qquad v = -4e^{-x/4}.$$

Accordingly,

$$\int x^2 e^{-x/4}\,\mathrm{d}x = -4x^2e^{-x/4} + 8\int xe^{-x/4}\,\mathrm{d}x.$$ *

To evaluate $\int xe^{-x/4}\,\mathrm{d}x$, we use Integration by Parts again with

$$u = x \qquad \text{and} \qquad \mathrm{d}v = e^{-x/4}\,\mathrm{d}x,$$ *

from which

$$\mathrm{d}u = \mathrm{d}x \qquad \text{and} \qquad v = -4e^{-x/4}.$$

Thus,

$$\int x^2 e^{-x/4}\,\mathrm{d}x = -4x^2e^{-x/4} + 8\left(-4xe^{-x/4} + \int 4e^{-x/4}\,\mathrm{d}x\right) \quad *$$

$$= -4x^2e^{-x/4} - 32xe^{-x/4} + 32\int e^{-x/4}\,\mathrm{d}x$$

$$= \boxed{-4x^2e^{-x/4} - 32xe^{-x/4} - 128e^{-x/4} + C} \quad *$$

6. To strip the root, we perform a trigonometric substitution with

$$x = 5\sec\theta \implies \mathrm{d}x = 5\sec\theta\,\tan\theta\,\mathrm{d}\theta\,. \quad *$$

When $x = 5$, $\theta = 0$; when $x = 10$, $\theta = \dfrac{\pi}{3}$. Thus, the integral becomes

$$\int_0^{\pi/3} \frac{(5\sec\theta)^3}{5\sqrt{25\sec^2\theta - 25}}(5\sec\theta\,\tan\theta)\,\mathrm{d}\theta = 125\int_0^{\pi/3}\frac{\sec^3\theta}{\sqrt{25\tan^2\theta}}(\sec\theta\,\tan\theta)\,\mathrm{d}\theta$$

$$= 25\int_0^{\pi/3}\sec^4\theta\,\mathrm{d}\theta\,. \quad *$$

The power of secant is even, so we conserve an extra factor $\sec^2\theta$ and convert the remaining secants to tangents:

$$25\int_0^{\pi/3}\sec^2\theta\,\sec^2\theta\,\mathrm{d}\theta = 25\int_0^{\pi/3}(\tan^2\theta + 1)\sec^2\theta\,\mathrm{d}\theta\,. \quad *$$

We strip the extra $\sec^2\theta$ by substituting

$$u = \tan\theta \implies \mathrm{d}u = \sec^2\theta\,\mathrm{d}\theta\,.$$

When $\theta = 0$, $u = 0$; when $\theta = \dfrac{\pi}{3}$, $\tan\theta = \sqrt{3}$. Accordingly, we attain

$$25\int_0^{\sqrt{3}}(u^2+1)\,\mathrm{d}u = 25\left(\tfrac{1}{3}u^3 + u\right)\Big|_0^{\sqrt{3}} \quad *$$

$$= \boxed{50\sqrt{3}} \quad *$$

7. When a polynomial is under a square root, it is a good idea to complete the square. Doing so produces

$$\int \frac{x+2}{\sqrt{30-8x-2x^2}}\,dx = \frac{1}{\sqrt{2}}\int \frac{x+2}{\sqrt{15-4x-x^2}}\,dx$$
$$= \frac{1}{\sqrt{2}}\int \frac{x+2}{\sqrt{19-(x+2)^2}}\,dx\,. \quad *$$

To strip the root sign, we perform a trigonometric substitution:

$$x+2=\sqrt{19}\sin\theta \implies dx=\sqrt{19}\cos\theta\,d\theta\,. \quad *$$

The integral therefore becomes

$$\frac{1}{\sqrt{2}}\int \frac{\sqrt{19}\sin\theta}{\sqrt{19-19\sin^2\theta}}(\sqrt{19}\cos\theta)\,d\theta = \frac{19}{\sqrt{2}}\int \frac{\sin\theta}{\sqrt{19\cos^2\theta}}(\cos\theta)\,d\theta \quad *$$
$$= \frac{\sqrt{19}}{\sqrt{2}}\int \sin\theta\,d\theta$$
$$= -\frac{\sqrt{19}}{\sqrt{2}}\cos\theta + C\,. \quad *$$

From $x+2=\sqrt{19}\sin\theta$, it follows by geometry that

$$\cos\theta = \frac{\sqrt{19-(x+2)^2}}{\sqrt{19}}\,.$$

Hence, the result is

$$-\frac{\sqrt{19}}{\sqrt{2}}\left(\frac{\sqrt{19-(x+2)^2}}{\sqrt{19}}\right) = \boxed{-\sqrt{\frac{19-(x+2)^2}{2}}+C} \quad *$$

8. We first substitute

$$w=\sqrt{x} \implies dw=\frac{1}{2\sqrt{x}}\,dx=\frac{1}{2w}\,dx\,. \quad *$$

Thus, $dx = 2w\,dw$. When $x=1$, $w=1$; when $x=4$, $w=2$. The integral therefore becomes

$$\int_1^2 e^w(2w)\,dw = \int_1^2 2we^w\,dw\,. \quad *$$

Now we use Integration by Parts with

$$u = 2w \qquad \text{and} \qquad \mathrm{d}v = e^w\,\mathrm{d}w\,.$$

Then

$$\mathrm{d}u = 2\,\mathrm{d}w \qquad \text{and} \qquad v = e^w\,.$$

Thus,

$$\int_1^2 2we^w = 2we^w\Big|_1^2 - \int_1^2 2e^w\,\mathrm{d}w \qquad *$$

$$= (2we^w - 2e^w)\Big|_1^2 \qquad *$$

$$= \boxed{2e^2} \qquad *$$

Long Questions (15 points each)

9. (a) In $\int \sec^3 x \tan^5 x \, dx$, the power of tangent is odd. We therefore conserve an extra factor $\sec x \tan x$ and convert the other factors to be in terms of secant:

$$\int \sec^2 x \tan^4 x \, (\sec x \tan x) \, dx = \int \sec^2 x \, (\sec^2 x - 1)^2 \, (\sec x \tan x) \, dx\,. \quad *$$

To strip the extra factor $\sec x \tan x$, we substitute

$$u = \sec x \implies du = \sec x \tan x \, dx\,. \quad *$$

So the integral becomes

$$\begin{aligned} \int \left[u^2(u^2-1)^2\right] du &= \int (u^6 - 2u^4 + u^2)\, du && * \\ &= \tfrac{1}{7}u^7 - \tfrac{2}{5}u^5 + \tfrac{1}{3}u^3 + C && * \\ &= \boxed{\tfrac{1}{7}\sec^7 x - \tfrac{2}{5}\sec^5 x + \tfrac{1}{3}\sec^3 x + C} && * \end{aligned}$$

(b) In $\int \sec^4 x \tan^2 x \, dx$, the power of secant is even. We therefore save an extra factor $\sec^2 x$ and express the other factors in terms of tangent:

$$\int \sec^2 x \tan^2 x \sec^2 x \, dx = \int (\tan^2 x + 1) \tan^2 x \sec^2 x \, dx\,. \quad *$$

We therefore substitute

$$u = \tan x \implies du = \sec^2 x \, dx\,. \quad *$$

Afterward, the integral becomes

$$\begin{aligned} \int \left[(u^2+1)\,u^2\right] du &= \int (u^4 + u^2)\, du && * \\ &= \tfrac{1}{5}u^5 + \tfrac{1}{3}u^3 + C && * \\ &= \boxed{\tfrac{1}{5}\tan^5 x + \tfrac{1}{3}\tan^3 x + C} && * \end{aligned}$$

(c) To clear the root, we substitute

$$x = 2\tan\theta \implies dx = 2\sec^2\theta \, d\theta\,. \quad *$$

The integral therefore becomes

$$\frac{1}{128}\int (2\tan\theta)^5\sqrt{4\tan^2\theta+4}\,(2\sec^2\theta)\,\mathrm{d}\theta = \frac{1}{128}\int (2\tan\theta)^5\sqrt{4\sec^2\theta}\,(2\sec^2\theta)\,\mathrm{d}\theta \quad *$$

$$= \int \tan^5\theta\sec\theta\sec^2\theta\,\mathrm{d}\theta \quad *$$

$$= \int \sec^3\theta\tan^5\theta\,\mathrm{d}\theta\,. \quad **$$

10. This is a rational expression whose denominator has a larger degree than the numerator, so integration by partial fractions is appropriate. The denominator is factored as follows:

$$x^4-81=(x+3)(x-3)(x^2+9)\,. \quad *$$

Hence, the shape of the partial fraction decomposition is

$$\frac{2x^2}{x^4-81}=\frac{A}{x+3}+\frac{B}{x-3}+\frac{Cx+D}{x^2+9}\,. \quad **$$

Multiplying both sides by $(x+3)(x-3)(x^2+9)$ then gives

$$2x^2 = A(x-3)(x^2+9)+B(x+3)(x^2+9)+Cx(x+3)(x-3)+D(x+3)(x-3)\,. \tag{1}$$

*

To solve for the coefficients, we perform a series of substitutions into Equation (1):

- *Substituting $x=3$.*

$$18 = 0+(6)(18)B+0+0 \implies B=\frac{1}{6}\,. \quad *$$

- *Substituting $x=-3$.*

$$18 = -(6)(18)A+0+0+0 \implies A=-\frac{1}{6}\,. \quad *$$

- *Substituting $x=0$.*

$$0=-\frac{1}{6}(-27)+\frac{1}{6}(27)-9D \implies D=1\,. \quad *$$

- *Substituting $x=1$ (arbitrary value).*

$$2=-\frac{1}{6}(-20)+\frac{1}{6}(40)-8C-8 \implies C=0\,. \quad *$$

Hence,

$$\int_0^1 \frac{2x^2}{x^4-81}\,\mathrm{d}x = \int_0^1 \left(\frac{-1/6}{x+3} + \frac{1/6}{x-3} + \frac{1}{x^2+9}\right)\mathrm{d}x. \tag{2}$$

To evaluate $\int \frac{1}{x^2+9}\,\mathrm{d}x$, we write

$$\int \frac{1}{x^2+9}\,\mathrm{d}x = \frac{1}{9}\int \frac{1}{\left(\frac{x^2}{9}\right)+1}\,\mathrm{d}x = \frac{1}{9}\int \frac{1}{\left(\frac{x}{3}\right)^2+1}\,\mathrm{d}x. \qquad *$$

We substitute

$$u = \frac{x}{3} \implies \mathrm{d}u = \frac{\mathrm{d}x}{3}. \qquad *$$

Thus,

$$\frac{1}{9}\int \frac{1}{\left(\frac{x}{3}\right)^2+1}\,\mathrm{d}x = \frac{1}{3}\int \frac{1}{u^2+1}\,\mathrm{d}u = \frac{1}{3}\tan^{-1}u = \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) \qquad *$$

(where the constant of integration is omitted). Consequently, Equation (2) becomes

$$\int_0^1 \frac{2x^2}{x^4-81}\,\mathrm{d}x = \left[-\frac{1}{6}\ln|x+3| + \frac{1}{6}\ln|x-3| + \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)\right]\Bigg|_0^1 \qquad **$$

$$= \left[\frac{1}{6}\ln\left|\frac{x-3}{x+3}\right| + \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)\right]\Bigg|_0^1$$

$$= \boxed{\frac{1}{6}\ln\left(\frac{1}{2}\right) + \frac{1}{3}\tan^{-1}\left(\frac{1}{3}\right)} \qquad **$$

11. (a) For $\int_{-1}^0 \frac{1-x}{2x^2-4x+6}\,\mathrm{d}x$, we substitute

$$u = 2x^2-4x+6 \implies \mathrm{d}u = (4x-4)\,\mathrm{d}x. \qquad *$$

With this substitution, we have $(1-x)\,\mathrm{d}x = -\frac{\mathrm{d}u}{4}$. When $x=-1$, $u=12$; when $x=0$, $u=6$. The integral therefore becomes

$$-\frac{1}{4}\int_{12}^{6}\frac{1}{u}\,\mathrm{d}u = \frac{1}{4}\int_{6}^{12}\frac{1}{u}\,\mathrm{d}u \qquad *$$

$$= \frac{1}{4}\ln|u|\Big|_{6}^{12} \qquad *$$

$$= \boxed{\frac{\ln 2}{4}} \qquad *$$

(b) In $\int_{-1}^{0}\frac{1-x}{2x^2+13x+6}\,\mathrm{d}x$, a substitution no longer works because the factor $(1-x)$ can no longer be stripped. Instead, we factor the denominator as

$$2x^2+13x+6 = (x+6)(2x+1)\,. \qquad *$$

The shape of the partial fraction decomposition is then

$$\frac{1-x}{2x^2+13x+6} = \frac{A}{x+6} + \frac{B}{2x+1}\,. \qquad *$$

Multiplying both sides by $(x+6)(2x+1)$ gives

$$1-x = A(2x+1)+B(x+6)\,.$$

Substituting $x=-6$ shows

$$7 = -11A+0 \implies A = -\frac{7}{11}\,.$$

Also, substituting $x=-\frac{1}{2}$ gives

$$\frac{3}{2} = \frac{11}{2}B \implies B = \frac{3}{11}\,.$$

Thus,

$$\int_{-1}^{0}\frac{1-x}{2x^2+13x+6}\,\mathrm{d}x = \int_{-1}^{0}\left(\frac{-7/11}{x+6}+\frac{3/11}{2x+1}\right)\mathrm{d}x \qquad *$$

$$= \int_{-1}^{0}\frac{-7/11}{x+6}\,\mathrm{d}x + \int_{-1}^{0}\frac{3/11}{2x+1}\,\mathrm{d}x\,.$$

The denominator of $\frac{3/11}{2x+1}$ is 0 at $x=-\frac{1}{2}$, so the graph of $y=\frac{3/11}{2x+1}$ has a vertical asymptote at

$x = -\frac{1}{2}$. Thus, $\int_{-1}^{0} \frac{3/11}{2x+1}\,\mathrm{d}x$ is improper, and we write

$$\int_{-1}^{0} \frac{3/11}{2x+1}\,\mathrm{d}x = \int_{-1}^{-1/2} \frac{3/11}{2x+1}\,\mathrm{d}x + \int_{-1/2}^{0} \frac{3/11}{2x+1}\,\mathrm{d}x$$

$$= \lim_{P\to(-1/2)^-} \int_{-1}^{P} \frac{3/11}{2x+1}\,\mathrm{d}x + \lim_{Q\to(-1/2)^+} \int_{Q}^{0} \frac{3/11}{2x+1}\,\mathrm{d}x\,. \quad *$$

The first integral becomes

$$\frac{3}{22} \lim_{P\to(-1/2)^-} \ln|2x+1| \Big|_{-1}^{P} = \frac{3}{22} \lim_{P\to(-1/2)^-} \ln|2P+1| - 0 = -\infty\,. \quad *$$

Because $\int_{-1}^{-1/2} \frac{3/11}{2x+1}\,\mathrm{d}x$ diverges, it follows that $\int_{-1}^{0} \frac{3/11}{2x+1}\,\mathrm{d}x$ also diverges and so

$$\int_{-1}^{0} \frac{1-x}{2x^2+13x+6}\,\mathrm{d}x\ \boxed{\text{diverges}} \quad *$$

(c) By completing the square, the integral becomes

$$\frac{1}{2}\int_{-1}^{0} \frac{(1-x)^3}{x^2-2x+3}\,\mathrm{d}x = \frac{1}{2}\int_{-1}^{0} \frac{(1-x)^3}{(x-1)^2+2}\,\mathrm{d}x\,. \quad *$$

With the substitution $x-1 = \sqrt{2}\tan\theta$, the differential is

$$\mathrm{d}x = \sqrt{2}\sec^2\theta\,\mathrm{d}\theta\,. \quad *$$

We change the bounds as follows:

- *Lower Bound*. When $x = -1$,

$$-2 = \sqrt{2}\tan\theta \implies \theta = \tan^{-1}(-\sqrt{2})\,.$$

- *Upper Bound*. When $x = 0$,

$$-1 = \sqrt{2}\tan\theta \implies \theta = \tan^{-1}\left(-\frac{1}{\sqrt{2}}\right)\,.$$

The integral therefore becomes

$$\frac{1}{2}\int_{\tan^{-1}(-\sqrt{2})}^{\tan^{-1}(-1/\sqrt{2})} \frac{(\sqrt{2}\tan\theta)^3}{2\tan^2\theta+2}(\sqrt{2}\sec^2\theta)\,\mathrm{d}\theta\,. \quad *$$

Because $2(\tan^2\theta+1)=2\sec^2\theta$, we have

$$\frac{1}{2}\int_{\tan^{-1}(-\sqrt{2})}^{\tan^{-1}(-1/\sqrt{2})} \frac{-2\sqrt{2}\tan^3\theta}{2\sec^2\theta}(\sqrt{2}\sec^2\theta)\,\mathrm{d}\theta = \boxed{-\int_{\tan^{-1}(-\sqrt{2})}^{\tan^{-1}(-1/\sqrt{2})} \tan^3\theta\,\mathrm{d}\theta} \quad **$$

12. (a) This is p-integral with $p=3c+6$. The integral converges when $p>1$:

$$3c+6>1 \quad *$$

$$\implies \boxed{c>-\frac{5}{3}} \quad *$$

(b) The exponent of x becomes 5. Now

$$\begin{aligned}
I &= \int_1^\infty \frac{1}{x^5}\,\mathrm{d}x & * \\
&= \lim_{t\to\infty}\int_1^t \frac{1}{x^5}\,\mathrm{d}x \\
&= \lim_{t\to\infty}\left(-\frac{1}{4x^4}\right)\Bigg|_1^t & * \\
&= \lim_{t\to\infty}\left(-\frac{1}{4t^4}\right)+\frac{1}{4} & * \\
&= 0+\frac{1}{4}=\boxed{\frac{1}{4}} & *
\end{aligned}$$

(c) Because $0\leqslant|\cos x|\leqslant 1$, we have

$$0\leqslant\frac{|\cos x|}{x^5}\leqslant\frac{1}{x^5}\,. \quad *$$

The improper integral $\int_1^\infty \frac{1}{x^5}\,\mathrm{d}x$ converges, so by comparison properties for improper integrals,

$$\int_1^\infty \frac{|\cos x|}{x^5}\,\mathrm{d}x \;\boxed{\text{converges}} \quad ***$$

(d) The power of x is $\frac{1}{3}$, so the integral is $\int_0^1 \frac{1}{\sqrt[3]{x}}\,\mathrm{d}x$. This integral is improper because the graph of the integrand has a vertical asymptote at $x = 0$, so

$$\int_0^1 \frac{1}{\sqrt[3]{x}}\,\mathrm{d}x = \lim_{t\to 0^+} \int_t^1 x^{-1/3}\,\mathrm{d}x \qquad *$$

$$= \lim_{t\to 0^+} \frac{3}{2}x^{2/3}\bigg|_t^1 \qquad *$$

$$= \frac{3}{2}(1)^{2/3} - \lim_{t\to 0^+} \frac{3}{2}t^{2/3} \qquad *$$

$$= \frac{3}{2} - 0 \qquad *$$

$$= \boxed{\frac{3}{2}} \qquad *$$